REPRESENTATION OF NON PERIODIC FUNCTIONS BY TRIGONOMETRIC SERIES WITH ALMOST INTEGER FREQUENCIES

GADY KOZMA AND ALEXANDER OLEVSKIĬ

Abstract. Inspired by Men'shov's representation theorem, we prove that there exists a sequence $\{\lambda(n)\} \subset \hat{\mathbf{R}}$, $\lambda(n) = n + o(1)$, $n \in \mathbf{Z}$ such that any measurable (complex valued) function f on $\mathbf{R}$ can be represented as a sum of almost everywhere convergent trigonometric series $\sum c_n e^{i\lambda(n)x}$.

1. Introduction

The Men'shov theorem (1940, see [Ba, ch. XV]) states that every measurable 2π-periodic f can be represented as a sum

$$f(x) = \sum_{n\in\mathbf{Z}} c_n e^{inx}$$

of (nonunique) trigonometric series convergent a.e. on $\mathbf{R}$. This famous result has served as a starting point for many further investigations, see [TO] for a comprehensive survey and references.

A non-periodic analog of the theorem is also known [D], where f is expanded in a "trigonometric integral" involving all frequencies $\lambda \in \hat{\mathbf{R}}$.

The aim of the present note is to show that by small perturbations of integers one can get a universal spectrum of frequencies which allows to represent any nonperiodic function on $\mathbf{R}$ by pointwise convergent trigonometric series.

Theorem. *There exists a real sequence* $\Lambda = \{\lambda(n)\}_{n=-\infty}^{\infty}$,

$$\lambda(n) = n + o(1) \tag{1}$$

such that every measurable function $f : \mathbf{R} \to \mathbf{C}$ *can be represented as a sum*

$$f(x) = \sum_{n\in\mathbf{Z}} c_n e^{i\lambda(n)x} \tag{2}$$

Partly supported by the Israel Science Foundation

convergent almost everywhere
Moreover, the perturbations $\alpha(n) = \lambda(n) - n$ can be obtained from an arbitrary pre-given sequence $0 \neq \rho(k) = o(1)$ ($k \in \mathbf{N}$) by rearrangement with finite repetitions.

Convergence is understood with respect to symmetrical partial sums, i.e. $\sum_{|\lambda(n)|<\eta} c_n e^{i\lambda(n)x}$, $\eta > 0$.

The idea of the proof below is to combine a recent result [Ol] about approximation by polynomials with “almost integer” frequencies with the elegant Körner version of Men'shov's technique, see [K].

Représentation de fonctions non-périodiques par des séries trigonométriques de fréquences presque entières.

Résumé. Inspirés par le théorème de représentation de Men'shov, nous prouvons qu'il existe une suite $\{\lambda(n)\} \subset \hat{\mathbf{R}}$, $\lambda(n) = n + o(1)$, $n \in \mathbf{Z}$ telle que toute fonction mesurable f peut être représentée comme somme de séries trigonométriques convergentes presque partout.

Version francaise abrégée. Le célèbre théorème de Men'shov affirme que toute fonction périodique mesurable f peut être représentée (de manière non-unique) comme somme de séries trigonométriques convergentes presque partout. Il y a de nombreuses versions et généralisations de ce résultat (voir [TO]). En particulier, un analogue nonperiodique de ce théorème est connu (voir [D]) dans lequel f est decomposée en "intégrale trigonométrique" utilisant toutes les fréquences. Le but de cet article est de montrer que par des petites perturbations d'entiers on peut obtenir un spectre universel des fréquences qui permet de représenter toute fonction non-périodique sur $\mathbf{R}$ par des séries trigonométriques convergentes point par point.

Théorème. *Il existe une suite réelle $\Lambda = \{\lambda(n)\}_{n=-\infty}^{\infty}$, $\lambda(n) = n + o(1)$ telle que toute fonction mesurable peut être représentée comme une somme $f(x) = \sum_{n\in\mathbf{Z}} c_n e^{i\lambda(n)x}$ convergente p.p.*

De plus, la perturbation peut être obtenue à partir d'une suite arbitraire $0 \neq \rho(k) = o(1)$ donnée par réarrangement avec un nombre fini de répétitions.

L'idée de la preuve ci-dessous est de combiner un résultat récent [Ol] sur l'approximation par des polynômes de fréquences presque entières avec la version élégante de Körner [K] des techniques de Men'shov.

2. Preliminaries

2.1. **Notations.** By a trigonometric polynomial we mean a finite sum

$$P(x) = \sum a_k e^{i\nu(k)x}, \quad \cdots \nu(-1) < \nu(0) < \nu(1) < \cdots \subset \mathbf{R}$$

The set $\{\nu(k)\}$ (the spectrum of P) is denoted by $\operatorname{spec} P$. When the $\nu(k)$ need to be integers, we shall specify so explicitly, by saying that the polynomial has integer spectrum. We will also use the following notations:

$$\begin{aligned} \deg P &= \max |\nu(k)| \\ ||P||_A &= \sum |a_k| \\ P^*(x) &= \sup_\eta \left| \sum_{|\nu(k)|<\eta} a_k e^{i\nu(k)x} \right| \\ ||P||_U &= \sup_{x\in\mathbf{R}} |P^*(x)| \\ P_{[N]}(x) &= P(Nx) \end{aligned}$$

$\mathbf{m}$ will denote the Lebesgue measure on the line.

2.2. **External lemmas.** We use the following known results. The first is lemma 1 from [K]

Lemma 1. *For every $\delta > 0$ there exists some constant $C(\delta)$ with satisfies the property that for every $\epsilon > 0$ there exists a trigonometric polynomial $P_{\epsilon,\delta}$ with integer spectrum with the following properties:*

1. $\widehat{P_{\epsilon,\delta}}(0) = 0$.
2. $\left|\widehat{P_{\epsilon,\delta}}(n)\right| < \epsilon \qquad \forall n.$
3. $\|P_{\epsilon,\delta}\|_U < C(\delta)$.
4. $\mathbf{m}\left(\{x \in [0, 2\pi] \;:\; |P_{\epsilon,\delta}(x) - 1| \geq \epsilon\}\right) < \delta$.

An inspection of Körner's proof of this lemma will show that it is possible to choose these polynomials such that if $\epsilon_1 \leq \epsilon_2$ and $\delta_1 \leq \delta_2$ then $\deg P_{\epsilon_1,\delta_1} \geq \deg P_{\epsilon_2,\delta_2}$. We shall denote $d(\epsilon, \delta) := \deg P_{\epsilon,\delta}$.

The next proposition is a slight variation of lemma 15 from [K].

Lemma 2. *Assume P, Q are trigonometric polynomials, P with integer spectrum, and N is a number satisfying $N > 2\deg Q$. Then*

$$\left(P_{[N]} \cdot Q\right)^* (x) \leq 2||\widehat{P}||_\infty \cdot ||Q||_A + |Q(x)| \cdot ||P||_U.$$

The last result is taken from [Ol]:

Lemma 3. *If $\sigma(n) = n+o(1)$, $\sigma(n) \neq n$, then the system $\left\{e^{i\sigma(n)x}\right\}_{n=-\infty}^{\infty}$ is complete in $L_0(\mathbf{R})$ i.e. for every measurable function $f : \mathbf{R} \to \mathbf{C}$ there exists polynomials $R_k = \sum c_{n,k} e^{i\sigma(n)x}$ such that $f(x) = \lim R_k(x)$ a.e. Further, this is true for any subsystem $\left\{e^{i\sigma(n)x}\right\}_{|n|>N}$.*

3. Proof of Theorem

3.1. **The sets I_l and the polynomials $R_{r,l}$.** Our first step is to define a fast-increasing sequence η_l such that the sets $I_l := [-\eta_l, -\eta_{l-1}] \cup [\eta_{l-1}, \eta_l]$ are "large enough" in the sense that the polynomials R_k in lemma 3 can be constructed each with the spectrum supported on one I_l.

More specifically, we shall construct polynomials $\{R_{r,l}\}_{0 \leq l,\, |r| \leq l}$, with spectrum in $\{\sigma(n)\}_{n=-\infty}^{\infty}$, $\sigma(n) := n + \rho(|n|)$ satisfying the following properties:

(1) $R_{r,l}$ have increasing spectra in the sense that if $l_1 > l_2$ or $l_1 = l_2$ and $r_1 > r_2$ then

$$\operatorname{spec} R_{r_1,l_1} \subset \{\xi \ :\ |\xi| > \deg R_{r_2,l_2}\}$$

(2) $R_{r,l}$ approximates $e^{i\sigma(r)x}$ in the sense

$$\mathbf{m}\left(\left\{x \in [-l\pi, l\pi] \ :\ |R_{r,l}(x) - e^{i\sigma(r)x}| \geq \frac{1}{l^2}\right\}\right) < \frac{1}{l^3} \tag{3}$$

The self evident induction using lemma 3 will yield these polynomials. We then define $\eta_l := \deg R_{l,l}$. Note that $\operatorname{spec} R_{r,l} \subset I_l$.

3.2. **Construction of Λ.** We first need auxiliary sequences of numbers. ϵ_l, $l \in \mathbf{N}$ will be a sequence decreasing so fast that

$$\epsilon_l \cdot \max_{|r| \leq l} ||R_{r,l}||_A < \frac{1}{l^2} \tag{4}$$

With ϵ_l we define $d_l := d(\epsilon_l, l^{-3})$ where $d(\epsilon, \delta)$ is defined after lemma 1. Finally we define b_l a sequence of integers satisfying $b_l > b_{l-1}d_{l-1} + \eta_{l-1} + 2\eta_l$. If we now define sets $I_{l,s} = I_l + sb_l$ then the sets $I_{l,s}$ will be disjoint for all l and $1 \leq |s| \leq d_l$. We can now define the sequence Λ on the union $J_l := \bigcup_{1 \leq |s| \leq d_l} I_{l,s}$ as follows:

$$\lambda(n + sb_l) = \sigma(n) + sb_l, \quad \sigma(n) \in I_l,\ 1 \leq |s| \leq d_l.$$

Λ is thus defined on $\{n : \sigma(n) \in \bigcup_l J_l\}$ and clearly satisfies (1) and that $\lambda(n) - n \in \{\rho(k)\}_{k=0}^{\infty}$. On the remaining n's Λ can be defined to be any arbitrary sequence satisfying these two conditions — for example $\lambda(n) = \sigma(n)$.

3.3. **Representation of f – definition of c_n.** Let now $f : \mathbf{R} \to \mathbf{C}$ be any measureable function. Our goal is to find coefficients c_n such that (2) holds a.e. We shall define successively both an increasing sequence $\{l(N)\}$ and blocks of coefficients corresponding to $J_{l(N)}$ and set $c_n = 0$ if $\sigma(n)$ does not belong to any $J_{l(N)}$. Thus

$$S_N := \sum_{j=1}^{N} \sum_{\sigma(n) \in J_{l(N)}} c_n e^{i\lambda(n)x}$$

would be a subsequence of partial sums of the series (2). Suppose that $N-1$ steps are already done, so we have S_{N-1}. We define $F_N := f - S_{N-1}$.

3.3.1. *First approximation – G_N.* We use lemma 3 to approximate F_N on the segment $[-N\pi\ N\pi]$ with a uniform error of $\delta_N = \frac{1}{NC((N+1)^{-3})}$ where the function $C(\delta)$ is taken from lemma 1; and a measure error of $\frac{1}{N^2}$ — namely, find a polynomial $G_N = \sum_{|r|<M_N} a_r e^{i\sigma(r)x}$ satisfying

$$\mathbf{m}\left(\{x \in [-N\pi, N\pi] \ : \ |G_N(x) - F_N(x)| \geq \delta_N\}\right) < \frac{1}{N^2} \tag{5}$$

3.3.2. *Second approximation – Q_N.* Now we choose the integer $l(N)$. We need it to be large enough, namely

$$l(N) > N,\ \delta_N^{-1},\ M_N,\ ||G_N||_A,\ l(N-1) \tag{6}$$

Q_N is then defined as follows:

$$Q_N = \sum_{|r|<M_N} a_r R_{r,l(N)}.$$

The estimate of $Q_N - G_N$ follows from (3):

$$\mathbf{m}\{x \in [-N\pi, N\pi] \ : \ |Q_N(x) - G_N(x)| \geq \delta_N\} \leq$$
$$\sum_{|r|\leq M_N} \mathbf{m}\left\{x \in [-N\pi\ N\pi] \ : \ |R_{r,l(N)}(x) - e^{i\sigma(r)x}| \geq \frac{\delta_N}{||G_N||_A}\right\} \leq$$
$$\sum_{|r|\leq M_N} \mathbf{m}\left\{x \in [-l(N)\pi, l(N)\pi] \ : \ |R_{r,l(N)}(x) - e^{i\sigma(r)x}| \geq \frac{1}{l^2(N)}\right\} \leq$$
$$\frac{2M_N+1}{l(N)^3} < \frac{3}{N^2} \tag{7}$$

Notice also that

$$\operatorname{spec} Q_N \subset I_{l(N)} \cap \{\sigma(n)\}_{n=-\infty}^{\infty}$$

and finally that

$$\text{(8)} \qquad ||Q_N||_A \leq ||G_N||_A \cdot \max_{|r|\leq l(N)} ||R_{r,l(N)}||_A.$$

3.3.3. *Third approximation – H_N.* The third approximation will come by multiplying Q_N with a certain polynomial. We first use lemma 1, with $\delta = N^{-3}$ and $\epsilon = \epsilon_{l_N}$. and get a polynomial P_N. Define

$$H_N := Q_N \cdot (P_N)_{[b_{l(N)}]} \quad .$$

Notice that $\deg P_N \leq d_{l(N)}$ and $\widehat{P_N}(0) = 0$. These two properties mean that $\operatorname{spec} H_N \subset J_{l(N)}$. Further, $\operatorname{spec} Q_N \subset \{\sigma(n)\}_{n=-\infty}^{\infty}$ implies that $\operatorname{spec} H_N \subset \Lambda$. This allows us to define

$$S_N := S_{N-1} + H_N.$$

3.4. **Convergence of (2).** We prove this in two stages: first that $S_N(x) \to f(x)$ a.e. and then that $H_N^*(x) \to 0$ a.e.

3.4.1. Lemma 1 clause 4 gives

$$\mathbf{m}\left(\left\{x \in [-N\pi, N\pi] \;:\; \left|(P_N)_{[b_{l(N)}]}(x) - 1\right| \geq \epsilon\right\}\right) < N\delta = \frac{1}{N^2}.$$

Now, on the "good" set of x such that $|P_N(b_{l(N)}x) - 1| < \epsilon$ we can use (4) and get

$$\text{(9)} \qquad \begin{aligned} |H_N(x) - Q_N(x)| &\leq ||Q_N||_A \cdot |P_N(b_{l(N)}x) - 1| \\ &\leq ||G_N||_A \cdot \max_{|r|\leq l(N)} ||R_{r,l(N)}||_A \cdot \epsilon_{l(N)} \\ &\leq ||G_N||_A \cdot \frac{1}{l(N)^2} \leq \frac{1}{l(N)} \leq \delta_N. \end{aligned}$$

Summing up (5), (7) and (9) we get for $f - S_N \equiv F_N - H_N$

$$\text{(10)} \qquad \mathbf{m}\left(\{x \in [-N\pi\; N\pi] \;:\; |f(x) - S_N(x)| \geq 3\delta_N\}\right) < \frac{5}{N^2}.$$

which implies $S_N(x) \to f(x)$ a.e. on $\mathbf{R}$.

3.4.2. Lemma 2 together with the estimate $||\hat{P}_N||_\infty < \epsilon_{l(N)}$ gives us

$$H_N^*(x) \le 2\epsilon_{l(N)} \cdot ||Q_N||_A + |Q_N(x)| \cdot ||P_N||_U \quad .$$

The first summand is $< \frac{1}{N}$ by (4), (6) and (8). Now, using (5) and (7) we get on $[-N\pi\, N\pi]$ minus a set of measure $\frac{4}{N^2}$ that

$$|Q_N(x)| \le |F_N(x)| + 2\delta_N \quad .$$

(10) for $N-1$ implies $|F_N(x)| \le 3\delta_{N-1} = \frac{3}{(N-1)C(N^{-3})}$ outside a set of measure $\frac{5}{(N-1)^2}$ and finally $||P_N||_U < C(N^{-3})$ so the second summand is $\le \frac{5}{N-1}$ on $[-N\pi\, N\pi]$ minus a set of measure $< \frac{9}{(N-1)^2}$. It follows that that $H_N^*(x) \to 0$ almost everywhere on $\mathbf{R}$. □

Remark. It might be interesting to compare the approximation and expansion results. The completeness theorem proved in [Ol] means that by *arbitrary small* perturbation of the integers one gets a spectrum Λ which is sufficient for approximation of any $f \in L^0(\mathbf{R})$ by linear combination of $e^{i\lambda x}$, $\lambda \in \Lambda$. In contrast to that in the expansion theorem the perturbations can not decrease fast. In particular, one can prove that in the theorem above it is impossible to construct the sequence Λ to satisfy the condition $\lambda(n) = n + O(n^{-\epsilon})$ for some $\epsilon > 0$.

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School of Mathematical Sciences, Tel Aviv University, Ramat-Aviv, Israel 69978

e-mail address: gady@math.tau.ac.il, olevskii@math.tau.ac.il