Stability of solutions to abstract differential equations

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Abstract

A sufficient condition for asymptotic stability of the zero solution to an abstract nonlinear evolution problem is given. The governing equation is $\dot{u} = A(t)u + F(t,u)$, where $A(t)$ is a bounded linear operator in Hilbert space H and $F(t,u)$ is a nonlinear operator, $\|F(t,u)\| \leq c_0\|u\|^{1+p}$, $p = const > 0$, $c_0 = const > 0$. It is not assumed that the spectrum $\sigma := \sigma(A(t))$ of $A(t)$ lies in the fixed halfplane $\text{Re}z \leq -\kappa$, where $\kappa > 0$ does not depend on t. As $t \to \infty$ the spectrum of $A(t)$ is allowed to tend to the imaginary axis.

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1 Introduction

Let H be a Hilbert space. Consider the problem

$$\dot{u} = A(t)u + F(t,u), \quad t \geq 0, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $\dot{u} = \frac{du}{dt}$ is the strong derivative, $A(t)$ is a linear closed densely defined in H operator with the domain $D(A)$, independent of t, $u_0 \in D(A)$. We assume that $F(t,u)$ is a nonlinear mapping, locally Lipschitz with respect to u, and satisfying the following inequality

$$\|F(t,u)\| \leq c_0\|u\|^{1+p}, \quad p > 0,\ c_0 > 0, \tag{3}$$

where p and c_0 are constants. We also assume that

$$\mathrm{Re}(Au, u) \le -\gamma(t)\|u\|^2, \quad \forall u \in D(A), \tag{4}$$

where

$$\gamma(t) > 0, \quad \lim_{t\to\infty} \gamma(t) = 0, \tag{5}$$

$$\gamma(t) = \frac{b_1}{(b_0 + t)^d}, \quad d = const \in (0, 1], \tag{6}$$

b_0 and b_1 are positive constants. Assumptions (5) are satisfied by the function (6). However, our method can be applied to many other $\gamma(t)$ satisfying assumptions (5).

Definition 1. *The zero solution to equation* (1) *is called Lyapunov stable if for any* $\epsilon > 0$*, sufficiently small, there exists a* $\delta = \delta(\epsilon) > 0$*, such that if* $\|u_0\| < \delta$*, then the solution to problem* (1) *exists on* $[0, \infty)$ *and* $\|u(t)\| \le \epsilon$*. If, in addition,*

$$\lim_{t\to\infty} \|u(t)\| = 0, \tag{7}$$

then the zero solution is asymptotically stable.

Basic results on the Lyapunov stability of the solutions to (1) one finds in [1]-[4], and in many other books and papers. In [4] these results are established under the assumption that the operator $A(t)$ is bounded, $D(A) = H$, and $A(t)$ has property $B(\nu, N)$. This means ([4], p.178) that every solution to the equation

$$\dot{u} = A(t)u \tag{8}$$

satisfies the estimate

$$\|u(t)\| \le Ne^{-\nu(t-s)}\|u(s)\|, \quad t \ge s \ge 0, \tag{9}$$

where $N > 0$ and $\nu > 0$ are some constants. The quantity

$$\kappa := \overline{\lim_{t\to\infty}} \frac{\ln \|u(t)\|}{t} \tag{10}$$

is called the exponent of growth of $u(t)$. If Σ is the set of κ for all solutions to (8), then

$$\kappa_s := \sup_{\kappa\in\Sigma} \kappa \tag{11}$$

is called senior exponent of growth of solutions to (8). The general exponent κ_g is defined as

$$\kappa_g := \inf \rho, \tag{12}$$

where ρ is the exponent in the inequality

$$\|u(t)\| \leq N e^{\rho(t-s)} \|u(s)\|, \quad t \geq s \geq 0. \tag{13}$$

One has

$$\kappa_s \leq \kappa_g, \tag{14}$$

and the case $\kappa_s < \kappa_g$ can occur (the Perron's example, see [4], p.177). If $\kappa_g < 0$ then the zero solution to (8) is Lyapunov asymptotically stable. If $A(t) = A$ does not depend on t and A is a bounded linear operator, then $\kappa_g < 0$ if and only if the spectrum of A, denoted $\sigma(A)$, lies in the halfplane $\mathrm{Re} z \leq \kappa_g < 0$. In this case

$$\|e^{At}\| \leq N_0 e^{\kappa_g t}, \tag{15}$$

and if $\|F(t,u)\| \leq q\|u\|$, $t \geq 0$, $\|u\| < \rho$, and $q < \frac{\kappa_g}{N_0}$, then equation (1) has negative general exponent also, so the zero solution to equation (1) is Lyapunov asymptotically stable ([4], p.403).

If $A = A(t)$, and for any solution to (8) estimate (9) holds with $\nu > 0$, and if (3) holds, then for any solution to (1) with $\|u_0\| \leq \delta$ and $\delta > 0$ sufficiently small, estimate (9) holds with a different $N = N_1$ and $\nu = \nu_1$, $0 < \nu_1 \leq \nu$ (see [4], p.414). This means that the zero solution to (1) is asymptotically stable under the above assumptions.

The basic new result of our work, Theorem 1 in Section 2, generalizes the above results to the case when the assumption $\kappa_g < 0$ is not valid. We allow the spectrum $\sigma(A(t))$ to approach imaginary axis as $t \to \infty$. This is the principally new generalization of the classical Lyapunov-Krein theory. If $\sqcap$ is the complex plane and l is the imaginary axis, then we assume that $\sigma(A(t)) \subset \sqcap$ for every $t \geq 0$, but we allow $\lim_{t\to\infty} d(\sigma(A(t)), l) = 0$, where $d(\sigma, l)$ is the distance between two sets σ and l. The new stability result is formulated in Theorem 1. In Lemma 1 an auxiliary result is formulated. A proof of Lemma 1 differs in details from the one in [7]. In Section 2 Theorem 1 and Lemma 1 are formulated. In Section 3 proofs are given. In Section 4 examples of applications of our method are given.

2 Formulation of the results

Lemma 1. *Let the inequality*

$$\dot{g}(t) \leq -\gamma(t) g(t) + a(t) g^{1+p}(t) + \beta(t), \tag{16}$$

hold for $t \in [0, T)$,*where* $g(t) \geq 0$ *has finite derivative from the right at every point* t *at which* $g(t)$ *is defined,* $\gamma(t) \geq 0$, $a(t) \geq 0$ *and* $\beta(t) \geq 0$ *are continuous on* $\mathbb{R}_+ := [0, \infty)$ *functions, and* $p = const > 0$. *Assume that there exists a* $\mu(t) \in C^1[0, \infty)$, $\mu(t) > 0$, $\dot{\mu}(t) \geq 0$, *such that*

$$a(t)[\mu(t)]^{-1-p} + \beta(t) \leq \mu^{-1}(t)[\gamma(t) - \dot{\mu}(t)\mu^{-1}(t)], \ t \geq 0, \qquad (17)$$

$$\mu(0)g(0) < 1. \qquad (18)$$

Then $g(t)$ *exists for all* $t \in [0, \infty)$ *and*

$$0 \leq g(t) < \mu^{-1}(t), \quad \forall t \geq 0. \qquad (19)$$

Theorem 1. *Assume that conditions* (1)-(6) *hold and* $b_1 > 0$ *is sufficiently large. Then the zero solution to* (1) *is asymptotically stable for any fixed initial data* $u(0)$.

3 Proofs

Proof of Lemma 1. Let $v(t) := g(t)e^{\int_0^t \gamma(s)ds} := g(t)q(t)$. Then (16) yields

$$\dot{v}(t) \leq q(t)a(t)q^{-(1+p)}(t)v^{1+p}(t) + q(t)\beta(t), \quad v(0) = g(0), \quad t > 0. \qquad (20)$$

We do not assume a priori that $v(t)$ is defined for all $t \geq 0$. This conclusion will follow from our proof. Denote $\eta(t) := q(t)\mu^{-1}(t)$, $\eta(0) = \mu^{-1}(0) > g(0)$. Using (18) and (20), one gets

$$\dot{v}(0) \leq a(0)v^{1+p}(0) + \beta(0) \leq \mu^{-1}(0)[\gamma(0) - \dot{\mu}(0)\mu^{-1}(0)] = \dot{\eta}(0). \qquad (21)$$

Since $v(0) = g(0) < \eta(0) = \mu^{-1}(0)$ by (18), and $\dot{v}(0) \leq \dot{\eta}(0)$, it follows that

$$v(t) < \eta(t), \quad 0 \leq t < \tau, \qquad (22)$$

where $\tau > 0$ is the right end of the maximal interval on which $v(t) < \eta(t)$, i.e., $\tau = \sup_{\{t \,:\, v(t)<\eta(t)\}} t$. Let us prove that $\tau = \infty$. Note that if (22) holds, then

$$\dot{v}(t) \leq \dot{\eta}(t), \quad 0 \leq t < \tau. \qquad (23)$$

Indeed, using (17) and (20) one obtains

$$\dot{v}(t) = q(t)(\dot{g} + \gamma g) \leq q(t)\mu^{-1}(t)[\gamma(t) - \dot{\mu}(t)\mu^{-1}(t)] = \dot{\eta}(t), \qquad (24)$$

as claimed. If $\tau < \infty$, then (22) and (23) imply

$$v(\tau - 0) - v(0) \leq \eta(\tau - 0) - \eta(0). \qquad (25)$$

Since $\eta(t) \in C^1[0,\infty)$ by definition, inequality (25) implies that $v(\tau-0) < \infty$ and, since $v(0) = g(0) < \mu^{-1}(0) = \eta(0)$, so that $v(0) < \eta(0)$, one gets

$$v(\tau - 0) < \eta(\tau - 0) < \infty. \tag{26}$$

Inequality (26) implies that $\tau = \infty$, because τ is the maximal interval $[0, \tau)$ of the existence of v, and if $\tau < \infty$ is the right end of the maximal interval of the existence of v then $\overline{\lim}_{t\to\tau-0} v(t) = \infty$, which contradicts (26). Thus, $\tau = \infty$ and, therefore, $T = \infty$.
Lemma 1 is proved. □

Proof of Theorem 1. Let $\|u(t)\| = g(t)$. Multiply (1) by $u(t)$, take the real part, and get

$$g(t)\dot{g}(t) \le -\gamma g^2(t) + c_0 g^{2+p}(t). \tag{27}$$

Since $g \ge 0$, inequality (27) is equivalent to

$$\dot{g}(t) \le -\gamma(t)g(t) + c_0 g^{1+p}(t). \tag{28}$$

If $g(t) > 0$, then (28) is obviously equivalent to (27). If $g(t) = 0$ $\forall t \in \Delta$, where $\Delta \subset \mathbb{R}_+$ is an open set, then $u(t) = 0$ $\forall t \in \Delta$, so $u(t) = 0$ $\forall t \ge 0$ by the uniqueness of the solution to the Cauchy problem for equation (1). This uniqueness holds due to the assumed local Lipschitz condition for F. If $g(t_0) = 0$, but $g(t) \ne 0$ for $(t_0, t_0+\delta)$ for some $\delta > 0$, then one divides (27) by $g(t)$ for $t \in (t_0, t_0+\delta)$, then one passes to the limit $t \to t_0+0$ and gets (28) at $t = t_0$. Let us explain the meaning of $\dot{g}(t_0)$ at a point where $u(t_0) = 0$. The function $\dot{u}(t)$ is continuous and it is known that $\frac{d\|u(t)\|}{dt} \le \|\dot{u}(t)\|$. We define $\dot{g}(t_0) = \lim_{s\to+0} \|u(t_0 + s)\| s^{-1}$. This limit exists and is equal to $\|\dot{u}(t_0)\|$. Choose

$$\mu(t) = \mu(0)e^{\frac{1}{2}\int_0^t \gamma(s)ds}, \quad \dot{\mu}(t)\mu^{-1}(t) = \gamma(t)/2. \tag{29}$$

Remark 1. *Note that* $\lim_{t\to\infty} \mu(t) = \infty$ *if and only if* $\int_0^\infty \gamma(t)dt = \infty$. *If* $\lim_{t\to\infty} \mu(t) = \infty$, *then* $\lim_{t\to\infty} \|u(t)\| = 0$. *Under the assumption* (6) *one has* $\int_0^\infty \gamma(t)dt = \infty$, *and we use this to derive some results about asymptotic stability. If* $d > 1$ *in* (6), *then* $\int_0^\infty \gamma(t)dt < \infty$, *and our methods can be used for a derivation of some results on stability, rather than asymptotic stability.*

Condition (18) is satisfied if

$$\mu(0) < [g(0)]^{-1}, \tag{30}$$

and we choose $\mu(0)$ so that this inequality holds. Using (29), one sees that inequality (17) is satisfied if

$$2c_0\mu^{-p}(0) \le \gamma(t)e^{\frac{p}{2}\int_0^t \gamma(s)ds}, \quad \forall t \ge 0. \tag{31}$$

Inequality (31) is satisfied if

$$2c_0\mu^{-p}(0) \le \gamma(0), \tag{32}$$

provided that

$$\gamma(0) \le \gamma(t)e^{\frac{p}{2}\int_0^t \gamma(s)ds} \qquad \forall t \ge 0. \tag{33}$$

Let us first use assumption (6) with $d \in (0,1)$:

$$\int_0^t \gamma(s)ds = b_1 \frac{(b_0+t)^{1-d} - b_0^{1-d}}{1-d}, \quad 0 < d < 1. \tag{34}$$

In this case $\gamma(0) = b_1 b_0^{-d}$, and inequality (33) holds if

$$2d < p b_1 b_0^{1-d}. \tag{35}$$

Inequality (35) is a sufficient condition for the function on the right of (33) to have non-negative derivative for all $t \ge 0$, i.e., to be monotonically growing on $[0,\infty)$, if $\gamma(t)$ is defined in (6). Conditions (32) and (35) hold if

$$2c_0\mu^{-p}(0) \le b_1 b_0^{-d} \quad \text{and} \quad 2d < p b_1 b_0^{1-d}. \tag{36}$$

For any fixed four parameters d, c_0, p, and $\mu(0) < [g(0)]^{-1}$, where $d \in (0,1)$, $c_0 > 0$, $p > 0$, and $\mu(0) > 0$, inequalities (36) can be satisfied by choosing *sufficiently large* $b_1 > 0$. With the choice of $\mu(t)$, given in (29), and the parameters $\mu(0)$, b_0 and b_1, chosen as above, one obtains inequality (19):

$$0 \le g(t) < \frac{e^{-\frac{b_1}{2(1-d)}[(b_0+t)^{1-d} - b_0^{1-d}]}}{\mu(0)}, \qquad d \in (0,1). \tag{37}$$

Since $g(t) = \|u(t)\|$, inequality (37) implies asymptotic stability of the zero solution to equation (1) for any initial value of u_0, that is *global* asymptotic stability. Moreover, (37) gives a rate of convergence of $\|u(t)\|$ to zero as $t \to \infty$.

Consider now the case $d = 1$, $\gamma(t) = b_1(b_0+t)^{-1}$,

$$\int_0^t \gamma(s)ds = b_1 \ln \frac{b_0+t}{b_0}, \qquad e^{\int_0^t \gamma(s)ds} = \left(\frac{b_0+t}{b_0}\right)^{b_1}. \tag{38}$$

In this case the choice of $\mu(t)$ in (29) yields

$$\mu(t) = \mu(0)\left(\frac{b_0+t}{b_0}\right)^{b_1/2}. \tag{39}$$

Choose $\mu(0)$ so that (30) holds, and fix it. Then inequality (31) holds if

$$2c_0\mu^{-p}(0) \le \frac{b_1}{b_0+t}\frac{(b_0+t)^{\frac{b_1 p}{2}}}{b_0^{\frac{b_1 p}{2}}}, \quad \forall t \ge 0. \tag{40}$$

Choose b_1 so that

$$b_1 p > 2, \qquad p > 0. \tag{41}$$

Then (40) holds if and only if it holds for $t = 0$, that is:

$$2c_0\mu^{-p}(0) \le \frac{b_1}{b_0}. \tag{42}$$

Inequality (42) is satisfied if either b_1 is chosen sufficiently large for any fixed b_0, or b_0 is chosen sufficiently small for any fixed $b_1 > 2p^{-1}$ (see (41)). In either case one concludes that the zero solution to equation (1) is globally asymptotically stable.

Theorem 1 is proved. □

4 Additional results. Examples

Example 1. Consider two equations:

$$\dot{u}(t) = Au(t), \tag{43}$$

$$\dot{v}(t) = Av(t) + B(t)v(t), \qquad t \ge 0, \tag{44}$$

where A and $B(t)$ are bounded linear operators in H, A *does not depend on* t, and

$$\int_0^\infty \|B(t)\|dt < \infty. \tag{45}$$

We assume that all the solutions to (43) are bounded. Then by the Banach-Steinhaus theorem the following inequality holds:

$$\sup_{t\ge0} \|e^{tA}\| \le c < \infty. \tag{46}$$

This implies Lyapunov's stability of the zero solution to (43), and the inclusion $\sigma(A) \subset \sqcap := \{z \ : \ \text{Re}z \le 0\}$, which implies $\text{Re}(Au, u) \le 0 \ \forall u \in H$. A well-known result is (see, e.g., [2]):

If (45) *and* (46) *hold then the zero solution to* (44) *is Lyapunov stable.*

The usual proof (see [2], where $H = \mathbb{R}^n$) is based on the Gronwall inequality. We give a new simple proof based on Lemma 1. Let $g(t) := \|v(t)\|$.

Multiply (44) by u, take the real part and use the inequality $\mathrm{Re}(Av, v) \leq 0$ to get: $g\dot{g} \leq \|B(t)\|g^2(t)$, $t \geq 0$. Using the inequalities $g(t) \geq 0$ and (45), one obtains

$$\dot{g}(t) \leq \|B(t)\|g(t), \quad g(t) \leq g(0)e^{\int_0^\infty \|B(s)\|ds} := c_1 g(0). \tag{47}$$

Therefore, the zero solution to (44) is Lyapunov stable. Moreover, since $|\dot{g}(t)| \in L^1(\mathbb{R}_+)$, it follows that there exists the finite limit: $\lim_{t\to\infty} \|v(t)\| := V$.

Example 2. Consider a theorem of N. Levinson in $\mathbb{R}^n$ (see [6] and [5], pp. 159-164):

If (45) *and* (46) *hold, then for every solution v to* (44) *one can find a solution u to* (43) *such that*

$$\lim_{t\to\infty} \|u(t) - v(t)\| = 0. \tag{48}$$

We give a new short proof of a generalization of this theorem to an infinite-dimensional Hilbert space H. If (45) and (46) hold, then, as we have proved in Example 1, $\sup_{t\geq 0} \|v(t)\| < \infty$, $\sup_{t\geq 0} \|u(t)\| < \infty$. If $u(0) = u_0$, then $u(t) = e^{tA}u_0$ solves (43). Let $v(t)$ solve the equation

$$v(t) = e^{tA}u_0 - \int_t^\infty e^{(t-s)A}B(s)v(s)ds. \tag{49}$$

A simple calculation shows that $v(t)$ solves (44) and

$$\|v(t)-u(t)\| \leq \int_t^\infty \|e^{(t-s)A}\|\|B(s)\|\|v(s)\|ds \leq C\int_t^\infty \|B(s)\|ds \to 0, \ t \to \infty, \tag{50}$$

where

$$C = \sup_{t\geq 0} \|e^{tA}\| \sup_{t\geq 0} \|v(t)\| < \infty.$$

The generalization of Levinson's theorem for H is proved. □

Equation (49) is uniquely solvable in H by iterations for all sufficiently large t because for such t the norm of the integral operator in (49) is less than one. The unique solution to (49) for sufficiently large t defines uniquely the solution v to (44) which satisfies (48).

Remark 2. *Our methods are applicable to the equation* (1) *with a force term:* $\dot{u} = A(t)u + F(t, u) + f(t)$.

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